Math 222A Lecture 4 Notes

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September 7, 2021

1 Continuous Dependence of ODEs on Initial Data and Classifications of PDEs

1.1 Continuous dependence of ODEs on initial data

Last time, we were discussing solving ODEs ofthe form

$$\begin{cases} u' = F(t,u) \\ u(0) = u_0. \end{cases}$$

We showed the following last time.

Theorem 1.1. *If F is locally Lipschitz, there exists a unique solution to the ODE.*

Today, we will talk more about continuous dependence of the solution on the initial data. So if we have $v' = F(t,v)$ with $v(0) = v_0$, we want to say that if $v(0)$ is close to $u(0)$, then v should be close to u.

Theorem 1.2. *Suppose that the solution u exists on $[0,T]$. Then there exists $\varepsilon > 0$ such that if $|v_0 - u_0| < \varepsilon$, then v exists on $[0,T]$ and*

$$\|u - v\|_C \leq c|u_0 - v_0|.$$

That is, the map $u_0 \mapsto u|_{[0,T]}$ is locally Lipschitz.

Proof. We compute

$$\begin{aligned} \frac{d}{dt}|u-v|^2 &= 2(u-v)\cdot(u-v)_t \\ &= 2(u-v)\cdot(F(u)-F(v)) \end{aligned}$$

If F is Lipschitz,

$$\leq 2L|u-v|^2.$$

So if $f(t) = |u-t|^2$, then $f'(t) \leq 2Lf(t)$ with $f(0) = |u_0 - v_0|^2$. We claim that this implies that $f(t) \leq f(0)e^{2Lt}$. This is called **Grönwall's inequality**.

Lemma 1.1 (Grönwall's inequality[1]). *If $f'(t) \leq 2Lf(t)$, then $f(t) \leq f(0)e^{2Lt}$.*

Proof. Let $g(t) = e^{-2Lt} f(t)$. It suffices to show that g is nonincreasing. We have $g'(t) = e^{-2Lt} f'(t) - 2Le^{-2Lt} f(t) \leq 0$. $\square$

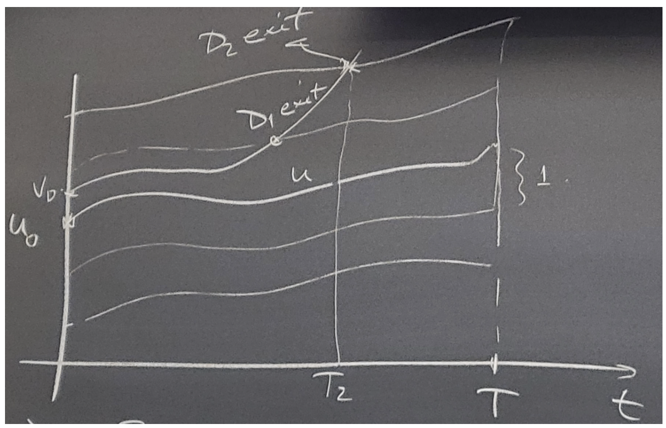
The proof is finished except for:

(a) If F is not globally Lipschitz.

(b) We do not know that v exists up to time T.

Suppose we have our solution u with initial data u. Consider two neighborhoods of u: a neighborhood $D_1 = \{v \in C([0,T]) : \|v - u\| \leq 1\}$ of size 1 and a neighborhood $D_2 = \{v \in C([0,T]) : \|v - u\| \leq 2\}$ of size 2.

Suppose we know that $v \in D_2$. Then v is defined on $[0,T]$, and stays in a compact set, so the above argument applies. How do we know v says in D_2? Suppose this is not true, so there is a time T_2 at which v exits D_2; then v must exit D_1 first.



By Grönwall's inequality applied to T_2, we have

$$\begin{aligned} |u(t) - v(t)|^2 &\leq |u_0 - v_0|^2 \cdot e^{2LT_2}, \qquad t \in [0, T_2] \\ &\leq \varepsilon^2 e^{2LT} \end{aligned}$$

Choosing ε sufficiently small,

$$\leq 1.$$

This implies that v does not exit D_1, which is a contradiction; to exit D_2, v must first exit D_1. $\square$

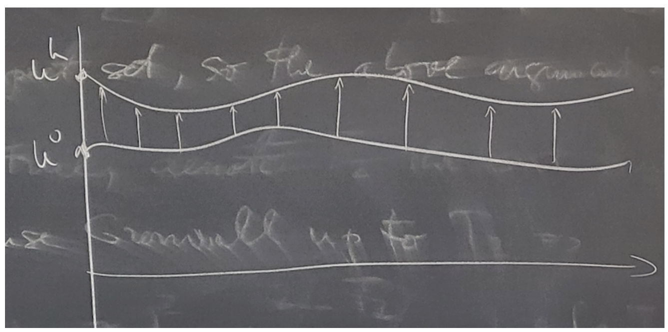
[1]More generally, we can prove this theorem with the same argument for $f'(t) \leq h(t)f(t)$.

Remark 1.1. Suppose we want to prove that if $\varepsilon \ll 1$, then $\|u - v\| \leq 1$. We made a **bootstrap assumption** $\|u - v\| \leq 2$ and used this assumption to prove $\|u - v\| \leq 1$. This is called a **bootstrap argument**. These kind of bootstrap arguments are useful in nonlinear PDEs, when you don't even know whether a solution exists.

1.2 Linearizing an equation

Assume $F \in C^1$ and suppose we have initial data u_0^0. Take a one-parameter family of data u_0^h with h close to 0, so this is differentiable in h. Let u_0^0 give a solution u^0 and u_0^h give a solution u^h. We can ask: how does u_h depend on h? We know that if $|u_0^h - u_0^0| \lesssim h$, then $|u^h - u^0| \lesssim he^{2LT}$ (with the notation $A \lesssim B$ meaning $A \leq cB$ for some constant c).

Here is a formal computation: If $\dot{u}^h = F(t, u^h(x))$, we want to compute an equation for $v^h = \frac{d}{dh}u^h$.



Apply $\frac{d}{dh}$ to get

$$\dot{v}^h = DF(t, u^h)v^h, \qquad v^h(0) = \frac{d}{dh}u_0^h.$$

This is a *linear* equation for v^h. It is called a **linearized equation**. This allows us to pass from one solution to another solution nearby.

Does the derivative actually exist? Let's compute:

$$\frac{d}{dt}(u^h - u^0) = F(t, u^h(T)) - F(t, u^0(t))$$

Think of this as a Taylor expansion

$$= DF(t, u^0(t))(u^h(t) - u^0(t)) + o(\underbrace{u^h(t) - u^0(t)}_{o(h)})^2.$$

Then

$$\frac{d}{dt}\frac{u^h - u^0}{h} = DF(t, u^0(t))\frac{u^h - u^0}{h} + o(h).$$

As $h \to 0$, $\frac{u^h - u^0}{h}(0) \to v^0$. So in the limit, we get $\frac{u^h - u^0}{h} \to v^0$, which is the solution to the linearized equation.

1.3 Classifications of first order scalar PDEs

We will study first order scalar PDEs. In these equations, we have $u : \mathbb{R}^n \to \mathbb{R}$, with

$$F(x, u, \partial u) = 0.$$

Evans' textbook uses Du instead of ∂u, but we will use this notation for something else later in the course.

Here is a classification by degree of difficulty:

- Linear:
$$\sum_j A_j(x)\partial_j u + B(x)u = f(x).$$
We can succinctly write this as $a \cdot \partial u + bu = f$.

- Semilinear:
$$\sum_j A_j(x)\partial_j u + b(x, u) = 0.$$
Here, the nonlinearity is only in u, not in the derivatives.

- Quasilinear:
$$\sum_j A_j(x, u)\partial_j u + b(x, u) = 0.$$

- Fully nonlinear:
$$F(x, u, \partial u) = 0.$$

If we differentiate a fully nonlinear PDE, we get a quasilinear PDE, but we get a system. For these equations, some things we know about scalar equations will not apply to systems.

What is our initial data? In $\mathbb{R}^n$, we take a surface Σ and specify $u|_\Sigma = u_0$ on the surface.

Definition 1.1. The equation plus our initial data is called an **initial value problem** or a **Cauchy problem**.

Another way we can classify partial differential equations is by static equations (at fixed time) and dynamic equations (evolution in time). This is a classification imposed less by the equations themselves and more by the motivation of the PDEs.

Example 1.1. The equation
$$u_t = F(x, u, \partial_x u)$$
with $u : \mathbb{R}_t \times \mathbb{R}_x \to \mathbb{R}$ is a dynamic or evolution equation. The **steady states** are solutions to the equation $0 = F(x, u, \partial_x u)$.

1.4 First order linear scalar PDEs

We are looking at the equation

$$\sum_j A_j(x) \cdot \partial_j u = bu + f,$$

which we can write as

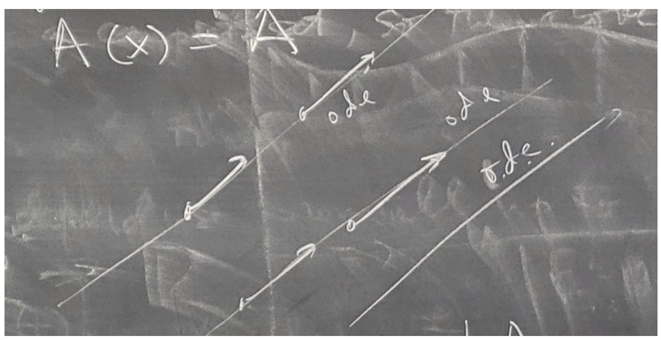
$$A \cdot \nabla u = bu + f,$$

where $A \cdot \nabla u$ is the directional derivative of u in the direction A.

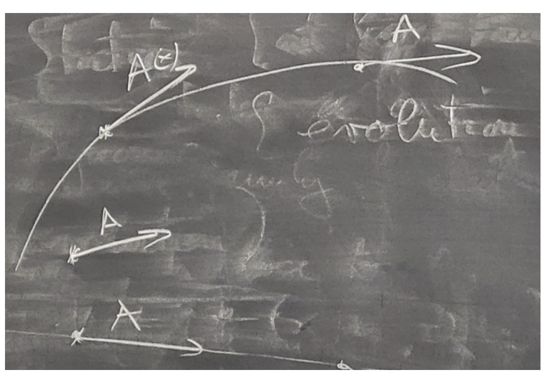
Let's start with a simpler case, where $A(x) = A$ does not depend on x. Then we can look at lines which point in the direction at A: $x = x_0 + tA$. Look at the function u along these lines: $u(x_0 + tA)$.

$$\begin{aligned} \frac{d}{dt} u(x_0 + tA) &= A \nabla u \\ &= bu(x_0 + tA) + f. \end{aligned}$$

This is a linear ODE for $u(x_0 + tA)$.



If A is not constant, can we do the same thing? Instead of straight lines, we need curves. In particular, we need curves which are tangent to A at each point.



Do such curves exist? The ODE $\dot{x}(t) = A(x(t))$ has C^1 solutions by ODE theory (where $A \in C^1$). So, given a point x, there is a unique curve starting from x that stays tangent to A. This is called an **integral curve** of A. We can calculate

$$\frac{d}{dt}u(x(t)) = \nabla u \cdot \dot{x}(t) = A\nabla u = bu(x(t)) + f,$$

which is an ODE for u. So if A is not constant, solving the PDE is like solving 2 ODEs: one that gives integral curves and one that tracks the solution u along each integral curve. Next time, we will look at what happens when we try to assign this initial data on a surface.